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# Remarks on geometric quantum mechanics

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## Abstract

Pursuing the aims of geometric quantum mechanics, it is shown in a geometrical fashion that, at least in finite dimension, Schrödinger dynamics enjoys classical complete integrability, and several consequences therefrom are deduced, including a Hannay-type reinterpretation of Berry's phase and a geometric description of some aspects of the quantum measurement problem.

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## 1. Introduction

In this note we wish to point out several classical features stemming from the very heart of the standard formalism of quantum mechanics. This is, in part, well known due to the work among others, of [4,10–16,25–27,40].

The basic idea of the geometric approach to quantum mechanics roughly consists in regarding it as classical mechanics on the projective Hilbert space associated to the quantum system, considered as given a priori, its dynamics being governed by a special class of Hamiltonians, namely those arising as mean values of self-adjoint operators (see Section 2).

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Our starting point is that, given such a Hamiltonian (confining ourselves to the finite dimensional non-degenerate case), there is a natural toral action leaving it invariant and foliating the projective space into Lagrangian (or isotropic) tori, thereby yielding complete integrability of the associated classical mechanical system (Section 3). The ensuing action-angle variables receive a natural interpretation, the former being, in particular, transition probabilities. This has been already shown in greater generality [16] using different techniques. Actually, the above theorem (in finite dimensions) can be also regarded as a consequence of a much more general result by Thimm [42] stating that  $U(n)$ - or  $O(n)$ -invariant Hamiltonian systems on symmetric spaces are completely integrable; furthermore, projective spaces provide the basic examples of Hamiltonian toric manifolds (see e.g. [5,21] or the textbooks [6,23,30]; also, for recent developments [34,35]). However, for the sake of definiteness, we give fully explicit arguments. In this way we possibly establish a link among different research strands.

Various implications of integrability—which do not seem to have been previously analyzed—are discussed in the subsequent sections.

First of all, it is natural to look anew at quantum adiabaticity and at the emergence of Berry's phase [7,39]: in view of classical complete integrability we can interpret this problem both quantum mechanically [7] and classical mechanically [24,32], showing compatibility of the two pictures (Section 4). Moreover it is interesting, in view of the statistical interpretation of quantum mechanics, to compute the partition function of the classical canonical ensemble explicitly (cf. [10–12]). This can be immediately achieved by resorting to the Duistermaat–Heckman formula [6,17,23,30] exploiting the toral action (Section 5; we sketch a direct elementary computation as well). In accordance with the suggestion of [10–12], we find that the partition function indeed differs from the standard quantum mechanical one by certain weights depending on the energy level spacings and reflecting the topological structure of the projective space as a  $CW$ -complex.

Furthermore (Section 6), we give a geometric interpretation of some aspects of the theory of quantum measurement (see e.g. [19,43] for recent surveys) in the version developed, e.g. in [9] (we stress the fact that we act within orthodox quantum mechanics). The passage from a pure state to a mixture after interaction with a measuring apparatus can be described in “classical” terms as averaging over the (“fast”) angle variables; one gets, as a by-product, a version of the averaging theorem (time averaging = angle averaging, [3]). The collapse of the wave function can also be described (though by no means “explained away”) by resorting to basic geometric invariant theory [22,33], by letting unitarity (but not linearity) be violated during the measurement process. The latter can be “visualized” geometrically in terms of a natural polytope (parametrizing toral orbits) emerging from convexity properties of the relevant moment map (cf. [5,6,21–23,30]).

Then, in Section 7, we show that second quantization can be realized via Bohr–Sommerfeld quantization (also cf. [4,38]). The final section summarizes our conclusions.

## 2. Review of geometric quantum mechanics

In this section we mostly follow and improve the treatment given in [40] (but see also [13–15,23]). Throughout the paper we assume  $\hbar = 1$ . Let  $V$  be a complex Hilbert space of

finite dimension  $n + 1$ , for simplicity, with scalar product  $\langle \cdot | \cdot \rangle$ , linear in the second variable. Let  $P(V)$  denote its associated projective space, of complex dimension  $n$ . This is the space of (pure) states in quantum mechanics. Upon free employ of Dirac’s bracket notation, we can identify a point in  $P(V)$ , which is, by definition, the ray (i.e. one-dimensional vector space)  $\langle v \rangle$  pertaining to (resp. generated by) a non-zero vector  $v \equiv |v\rangle$ —and often conveniently denoted by  $[v]$ —with the projection operator onto that line, namely

$$[v] = \frac{|v\rangle\langle v|}{\|v\|^2} \tag{2.1}$$

(actually, the above identification can be interpreted in terms of the moment map defined below). For the sequel, we notice that, upon choosing an orthonormal basis  $(e_0, e_1, \dots, e_n)$  of  $V$ , and setting, for a unit vector  $v = \sum_{i=0}^n \alpha_i e_i$ , the above projection can be written as a density matrix ([9,31], see also Section 6)

$$|v\rangle\langle v| \leftrightarrow (\bar{\alpha}_i \alpha_j) \tag{2.2}$$

(with  $\sum_{i=0}^n |\alpha_i|^2 = 1$ ). If  $U(V)$  denotes the unitary group pertaining to  $V$ , with Lie algebra  $u(V)$ , consisting of all skew-hermitian endomorphisms of  $V$ —which we call observables, with a slight abuse of language—then the projective space  $P(V)$  is a  $U(V)$ -homogeneous Kähler manifold. The isotropy group (stabilizer) of a point  $[v] \in P(V)$  is isomorphic to  $U(V') \times U(1)$ , with  $V'$  the orthogonal complement to  $\langle v \rangle$  in  $V$ , the  $U(1)$  part coming from phase invariance:  $[e^{i\alpha}v] = [v]$ . Hence

$$P(V) \cong \frac{U(V)}{(U(V') \times U(1))} \cong \frac{U(n + 1)}{U(n) \times U(1)}. \tag{2.3}$$

The fundamental vector field  $A^\sharp$  associated to  $A \in u(V)$  reads (evaluated at  $[v] \in P(V)$ ,  $\|v\| = 1$ )

$$A^\sharp|_{[v]} = |v\rangle\langle Av| + |Av\rangle\langle v|. \tag{2.4}$$

In view of homogeneity, these vectors span the tangent space of  $P(V)$  at each point. The (action of the) complex structure  $J$  reads, accordingly:

$$J|_{[v]}A^\sharp|_{[v]} = |v\rangle\langle iAv| + |iAv\rangle\langle v|. \tag{2.5}$$

Next we are going to write down the expression for the natural (i.e. Fubini-Study) metric  $g$  and Kähler form  $\omega$  (recalling that, if  $\text{Tr}$  denotes the trace on  $\text{End}(V)$ , then clearly  $\text{Tr}(|v\rangle\langle w|) = \langle w|v\rangle$ ): they are essentially the real and imaginary part (respectively) of the hermitian form  $\langle dv|dv\rangle$ . Explicitly:

$$g_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = \text{Re}\{\langle Av|Bv\rangle + \langle v|Av\rangle\langle v|Bv\rangle\}, \tag{2.6}$$

and

$$\omega_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = g_{[v]}(J|_{[v]}(A^\sharp|_{[v]}), B^\sharp|_{[v]}) = \frac{i}{2}\langle v|[A, B]v\rangle. \tag{2.7}$$

Actually, our discussion can be conveniently rephrased in terms of the *moment map*

$$\mu : P(V) \rightarrow u(V)^* \cong u(V), \quad \mu([v]) = -i|v\rangle\langle v| \tag{2.8}$$

(the last isomorphism coming from the Killing–Cartan metric on  $u(V)$  given by  $(A, B) := -(1/2)\text{Tr}(AB)$ , for  $A, B \in u(V)$ ). The Hamiltonian algebra corresponding to  $\mu$  consists, accordingly, of the real smooth functions

$$\mu_A([v]) = (\mu, A) = \frac{i}{2} \langle v|Av \rangle, \quad A \in u(V), \tag{2.9}$$

i.e., up to a constant, the mean values of the observables. It follows immediately that  $\omega$  emerges as the canonical Kirillov symplectic form pertaining to  $P(V)$  looked upon (via  $\mu$ ) as a  $U(V)$ -coadjoint orbit (see e.g. [23,30]). Clearly,  $A^\sharp$  becomes the Hamiltonian vector field associated to  $A \in u(V)$ , i.e. one has

$$d\mu_A = i_{A^\sharp}\omega. \tag{2.10}$$

The Poisson bracket  $\{\cdot, \cdot\}$  defined by  $\omega$  is of course

$$\{\mu_A, \mu_B\} := \omega(A^\sharp, B^\sharp) = \mu_{[A, B]}. \tag{2.11}$$

We also notice, for further use, that for  $A, B \in u(V)$ , one has

$$[A^\sharp, B^\sharp] = -[A, B]^\sharp, \tag{2.12}$$

where the l.h.s. commutator refers to vector fields, the r.h.s. one is the Lie algebraic one. The latter identity can be directly checked by evaluating both sides on a Hamiltonian  $\mu_C$ .

From this point of view we may characterize Fubini-Study Killing vector fields as the infinitesimal generators of unitary one-parameter groups, i.e., with the Hamiltonian vector fields  $A^\sharp$  (also cf. [13]).

We now wish to compute a (local) symplectic potential  $\theta$  for  $\omega$ , i.e. a one-form such that  $d\theta = \omega$ . The one-form  $\theta$  cannot be global since a symplectic form on a compact manifold cannot be exact: indeed, it generates the one-dimensional second cohomology group  $H^2(P(V))$  and gives rise to the first Chern class of the hyperplane section bundle  $\mathcal{O}(1)$ , whose space of holomorphic sections is canonically (conjugate linear) isomorphic to  $V$  (see also Section 7, and [20]).

A short computation involving (2.12) shows that we may take

$$\theta = -i \langle v|dv \rangle. \tag{2.13}$$

Up to a constant,  $\theta$  is just the canonical (Chern–Bott) connection form (with respect to a hermitian local frame) on  $\mathcal{O}(1)$ , governing the so-called Berry (or, rather Aharonov–Anandan) phase ([1,2,7,20], see also Section 4). Geometrically, it just represents the infinitesimal angle variation of  $v$  (relative to the complex plane it generates) upon an infinitesimal (norm-preserving) displacement. This will be crucial for the sequel.

### 3. Toral actions and integrability

Let us now start from a non-degenerate quantum Hamiltonian

$$H = \sum_{j=0}^n \lambda_j P_j = \sum_{j=0}^n \lambda_j |e_j\rangle \langle e_j|, \tag{3.1}$$

i.e.  $\lambda_i \neq \lambda_j$ , if  $i \neq j$ , and  $(e_j)$  is an orthonormal basis of eigenvectors, with  $P_j := |e_j\rangle\langle e_j|$  being the orthogonal projection operator onto the line  $\langle e_j|$ . Without loss of generality we assume  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n$ , so

$$H = \sum_{j=1}^n \lambda_j P_j. \tag{3.2}$$

The Schrödinger evolution is given by (recall that  $\hbar = 1$ )

$$\frac{\partial}{\partial t} |v\rangle = -iH|v\rangle, \tag{3.3}$$

inducing its projective space version ([10–12,26], in which the spinor formalism is used):

$$\left| \frac{\partial}{\partial t} v \right\rangle \langle v| + |v\rangle \left\langle \frac{\partial}{\partial t} v \right| = i|Hv\rangle\langle v| - i|v\rangle\langle Hv| \tag{3.4}$$

(here  $\|v\| = 1$ ). Its mean value on a state  $[v]$  yields a “classical” Hamiltonian  $h$  on  $P(V)$ ; with the above notations

$$h([v]) = \frac{\langle v|Hv\rangle}{\langle v|v\rangle} = \frac{\sum_{j=0}^n \lambda_j |\alpha_j|^2}{\sum_{j=0}^n |\alpha_j|^2} = \sum_{j=1}^n \lambda_j |\alpha_j|^2, \tag{3.5}$$

the last equality holding for  $\|v\| = 1$ ,  $\lambda_0 = 0$ . Comparison with (2.9) yields

$$h([v]) = \mu_{(-2iH)}. \tag{3.6}$$

The critical points of  $h$  are given by the zeros of  $(-iH)^\sharp$  (symplectic gradient) or equivalently  $J(-iH)^\sharp = H^\sharp$  (Riemannian gradient), and these, in turn correspond to the states  $[e_j]$  determined by the eigenvectors  $e_j$ . This can be seen in various ways, for instance via the immediately checked formula for the dispersion (variance) of an observable  $A \in u(V)$  in a state  $[v]$ , see e.g. [2,15,40]:

$$\Delta_{[v]} A = \|Av - \langle v|Av\rangle v\| = \|A^\sharp_{[v]}\|_{FS} := \sqrt{g_{[v]}(A^\sharp_{[v]}, A^\sharp_{[v]})} = \|J_{[v]} A^\sharp_{[v]}\|_{FS}. \tag{3.7}$$

The nature of the critical point  $[e_j]$  can be ascertained via the formula (resorting to normalized vectors and then to obviously defined real coordinates)

$$h([v]) = \lambda_j + \sum_{k=0}^n (\lambda_k - \lambda_j) |\alpha_k|^2 = \lambda_j + \sum_{k=0}^n (\lambda_k - \lambda_j) (x_k^2 + y_k^2), \tag{3.8}$$

showing, in particular, that  $h$  is a perfect Morse function, i.e. the index of the  $j$ th critical point, namely  $2j$ , yield the Betti number  $b_{2j}(P(V)) = 1$  (the odd ones vanish).

Now let  $v = \sum_{j=0}^n \alpha_j e_j$ , with  $\alpha_j \neq 0$  for all  $j = 0, \dots, n$ . The submanifold consisting of such  $[v]$ 's is open and dense in  $P(V)$ . The torus  $\mathbf{T}^{n+1}$  acts on  $P(V)$  via the position  $e_j \mapsto e^{i\beta_j} e_j$ ,  $\beta_j \in [0, 2\pi)$ , but actually, in view of global phase arbitrariness this action descends to an effective action of  $G := \mathbf{T}^n$ : this is clearly seen in the density matrix formalism

$$(\bar{\alpha}; \alpha_j) \mapsto (\bar{\alpha}; \alpha_j e^{i(\beta_j - \beta_i)}) \tag{3.9}$$

(we shall resume this particular formalism in Section 6). We set  $\beta_0 = 0$  in order to be specific. The generators of the torus action are the (mutually commuting) operators  $iP_j$ ,  $j = 1, 2, \dots, n$ . Their associated Hamiltonians  $p_j := \langle \cdot | P_j \cdot \rangle = \mu_{(-2iP_j)}$  give rise to  $n$  constants of motion (first integrals) in involution, with respect to the Poisson bracket (2.12) defined by the Fubini-Study form, which turn out to be the action variables (see below). In the complement we have a stratification of toral orbits of dimensions  $k = 0, 1, \dots, n - 1$  (isotropic tori), but the basic picture persists. Precisely, we may state the following theorem.

**Theorem 3.1.**

- (i) Under the above assumptions, the “classical” Hamiltonian system  $(P(V), \omega, h)$  (actually an open dense set thereof) is completely integrable. The Lagrangian tori are provided by the orbits  $G \cdot [v]$  of the  $n$ -dimensional torus  $G$ -action above. The action variables  $I_j$  coincide with the transition probabilities  $|\alpha_j|^2 = p_j([v])$ ,  $j = 1, 2, \dots, n$ .
- (ii) Indeed, the full system remains integrable, allowing isotropic tori, and the orbit space can be identified with the standard  $n$ -simplex in the Euclidean space  $\mathbf{R}^n$ .

**Proof.** Ad (i). We compute the action variables  $I_j$ ,  $j = 1, 2, \dots, n$  in the standard fashion [3].

If  $\vartheta$  is a (local) potential of the symplectic form, they read, upon choosing a homology basis  $(\gamma_j)$  for a fixed Lagrangian torus

$$I_j = \frac{1}{2\pi} \int_{\gamma_j} \vartheta. \tag{3.10}$$

In our case, considering a generic orbit  $G \cdot [v]$  (which is topologically an  $n$ -dimensional torus itself and it is clearly Lagrangian, since  $\omega|_{G \cdot [v]} \equiv 0$ ) we may take as  $\gamma_j$  the curves

$$[0, 2\pi) \ni \beta_j \mapsto \left[ \sum_{h \neq j} \alpha_h e_h + \alpha_j e^{i\beta_j} e_j \right] \in P(V) \tag{3.11}$$

and, recalling (2.13), we easily get

$$I_j = \frac{1}{2\pi} \int_0^{2\pi} |\alpha_j|^2 (-i(e^{i\beta_j} e_j | d e^{i\beta_j} e_j)) = |\alpha_j|^2. \tag{3.12}$$

The Schrödinger evolution reads, in coordinates (taking as before  $\lambda_0 = 0$ ):

$$v = \sum_{i=0}^n \alpha_i e_i \mapsto \sum_{i=0}^n \alpha_i e^{-i\lambda_i t} e_i = \alpha_0 e_0 + \sum_{i=1}^n \alpha_i e^{-i\lambda_i t} e_i, \tag{3.13}$$

and induces an obvious evolution on the torus  $G \cdot [v]$ .

Ad (ii). The action variables  $I_j$ ,  $j = 1, 2, \dots$ , are globally defined, and collectively they give rise to the convex polytope (in  $\mathbf{R}^n$ )

$$0 \leq \sum_{j=1}^n I_j = 1 - |\alpha_0|^2 \leq 1, \tag{3.14}$$

which is actually the standard  $n$ -simplex  $\Delta_n$  in  $\mathbf{R}^n$ . Thus, the orbit space is just  $\Delta_n$ , the singular  $k$ -toral orbits,  $0 \leq k < n$  corresponding to its  $k$ -faces.  $\square$

### Remarks.

1. As we have already pointed out, this result is known in different guises, though possibly not so directly (cf. [5,16,21,23,30]). This concerns, in particular, the identification of action variables with transition probabilities, which is important for the sequel.
2. The geometry of the energy surfaces alone is quite intricate [12]; in the latter paper it has been observed that Schrödinger evolution takes place on a torus, but apparently there is no mention of integrability.

## 4. Berry and Hannay angles

In this section we wish to reinterpret the emergence of Berry's geometric phase [7,8,39] after cyclic adiabatic perturbations of the Hamiltonian within the classical interpretation of the quantum mechanical formalism outlined in the previous section. In an adiabatic evolution of a non-degenerate Hamiltonian  $H = H(R)$  depending on a point  $R \in \mathcal{R}$  (parameter space, of dimension  $\geq 2$ ) eigenvectors evolve into eigenvectors (see e.g. [9,31] and particularly [39] for a careful discussion of the "quantum adiabatic theorem") and, if the evolution is also cyclic, a final eigenvector differs from the initial one by a phase factor (Berry's phase), which can be ascribed to parallel transport via the Chern–Bott connection on  $\mathcal{O}(1)$  (cf. Section 2). In what follows we shall neglect the so-called dynamical phase. Explicitly, if  $C : [0, T] \rightarrow \mathcal{R}$  denotes a closed oriented circuit in the parameter space

$$e_j(C(T)) = e^{i \int_C -i(e_j(R)|d_{\mathcal{R}}e_j(R))} \cdot e_j(C(0)) =: e^{i\Delta\vartheta_j^B} \cdot e_j(C(0)) \quad (4.1)$$

since, again by the very definition of the Chern–Bott connection, the infinitesimal angle variation, say  $d\vartheta_j$ , of  $e_j(R)$  in the complex plane in  $V$  it determines is  $-i(e_j(R)|d_{\mathcal{R}}e_j(R))$  (the differential being now taken with respect to the parameter space  $\mathcal{R}$ , pulling back everything from  $P(V)$  to  $\mathcal{R}$ ). We have tacitly assumed that in our evolution  $e_0(C(t)) \equiv e_0$  for all  $t \in [0, T]$ .

Now, the adiabatic perturbation induces a *migration* of the Lagrangian tori (and isotropic ones) pertaining to the quantum system, happening on the trivial fibration  $\mathcal{R} \times P(V) \rightarrow \mathcal{R}$ : but this is exactly the framework leading to the appearance of Hannay's angles [24,32]; the migration is governed by Montgomery's connection (given by averaging over tori [32]). We wish to show that the two pictures are compatible: upon computing the relevant Hannay's angles  $\Delta\vartheta_j^H$ , we shall recover Berry's phases  $\Delta\vartheta_j^B$ .

**Theorem 4.1.** *With the above notations, we have*

$$\Delta\vartheta_j^H = \Delta\vartheta_j^B, \quad j = 1, 2, \dots, n. \quad (4.2)$$

**Proof.** Averaging  $d\vartheta_j$  over the torus  $G$ , with respect to its normalized Haar measure  $dg$ , leaves it unaltered:  $\langle d\vartheta_j \rangle_G = d\vartheta_j$  by virtue of its geometric significance. The full Hannay

angle  $\Delta\vartheta_j^H$  is obtained by integrating along the closed oriented circuit  $C$  in the parameter space, yielding

$$\Delta\vartheta_j^H = \int_C (-i\langle e_j(R) | d_{\mathcal{R}} e_j(R) \rangle) = \Delta\vartheta_j^B, \quad j = 1, 2, \dots, n. \quad \square \quad (4.3)$$

The geometric phase phenomenon can be synthetically described via the density matrix formalism (cf. Eq. (3.9)). The off-diagonal (interference) terms have been detected in specific experiments.

**Remark.** Note that in Theorem 4.1 we looked upon the same (quantum) system in two different ways. This is different from Berry’s analysis describing the relationship between his and Hannay’s angles via a semiclassical analysis of an integrable system [8].

### 5. The partition function

In classical statistical mechanics it is natural to consider, among others, the canonical ensemble partition function pertaining to a classical Hamiltonian system. This is particularly relevant in our case in view of the statistical interpretation of quantum mechanics; so in this section we are going to compute the canonical ensemble partition function  $Z = Z(\beta)$  associated to the Hamiltonian system  $(P(V), \omega, h)$  explicitly, slightly improving some results of [10–12]. Recall that

$$Z(\beta) = \int_{P(V)} e^{-\beta h(v)} \frac{1}{n!} \omega^n, \quad (5.1)$$

where  $\beta \in \mathbf{R}$  (actually, the formula holds for  $\beta$  complex, after suitable interpretation), and  $\omega$  is, in this section, *one-half* of the previous one, in order to comply with the convention adopted in [30]. This particular issue has been addressed in [10–12], but apparently they give no explicit formulae (except for  $n = 1$  in a slightly different context) in addition to qualitative remarks. So we resort to the Duistermaat–Heckman formula [17,22,30], concerning exactness of the stationary phase approximation, and we also provide a (sketch of a) direct calculation. This is possible in view of the toral invariance of the classical Hamiltonian  $h$  (cf. formula (3.5)); proper handling of square roots eventually yields the following.

**Theorem 5.1.** *The partition function  $Z$  pertaining to  $P(V)$  and to the  $G$ -invariant Hamiltonian  $h$  reads, explicitly*

$$Z(\beta) = \left(\frac{\pi}{\beta}\right)^n \sum_{j=0}^n w_j e^{-\beta\lambda_j}, \quad (5.2)$$

where

$$w_j = \prod_{i \neq j} (\lambda_i - \lambda_j)^{-1}. \quad (5.3)$$



**Sketch of proof.** The proof is just a matter of tracing back the very definitions and appropriate conventions (see e.g. [22,30]). Actually, it is also possible to give an elementary computation, outlined below. We stick to the case  $n = 1$  for simplicity, the general case being dealt with similarly. First, realize  $P(\mathbf{C}) \cong S^2 \cong S^3/S^1$ . Due to the manifest  $S^1$ -invariance of  $Z$ , it is enough to compute on  $S^3$  and then divide by  $2\pi$  (notice that the area of  $P(\mathbf{C})^2$  is  $\pi$ ). Now  $S^3$  is described by the standard embedding in  $\mathbf{R}^4$ , i.e. (obvious notation)

$$S^3 = \{(x_0, y_0, x_1, y_1) | x_0^2 + y_0^2 + x_1^2 + y_1^2 = 1\}. \tag{5.4}$$

The volume form  $d\text{vol}_{S^3}$  reads (after suitable arrangement)

$$d\text{vol}_{S^3} = (x_0 dy_0 - y_0 dx_0) \wedge dx_1 \wedge dy_1 + (x_1 dy_1 - y_1 dx_1) \wedge dx_0 \wedge dy_0. \tag{5.5}$$

Passing to polar coordinates (in the appropriate planes:  $z_j = \varrho_j e^{i\theta_j}$ ), the integral becomes a sum of two contributions. In the first, we rewrite the exponential as  $-\beta[\lambda_0 + (\lambda_1 - \lambda_0)\varrho_1^2]$ , and perform a similar trick for the second piece. Integrating over angles we are left with simple  $\varrho$ -integrals, which finally yield (partial cancellations occurring) the formula for  $Z$  in this case.

We notice that, as a retrospective check, one gets  $\lim_{\beta \rightarrow 0} \text{r.h.s.} = \pi^n/n! = \text{vol}(P(V))$ .

So, following [12], we may assert that the canonical partition function differs from the standard quantum mechanical one in that the presence of the weights  $w_j$  encodes information about energy level spacings, this being related to the Hessian of the Hamiltonian at critical points, which, in turn, is related to the topology of  $P(V)$  as a CW-complex via Morse theory. Recall that  $P(V)$  is made up of  $2k$ -dimensional cells, one for each  $k = 0, \dots, n$ , this being also reflected by the (de Rham) cohomology algebra, which is generated by the Fubini-Study form, whose various exterior products yield the appropriate Poincaré–Cartan invariants (see e.g. [20]).

It has been advocated in [10,12] that this “classical” partition function is more natural than the standard quantum mechanical one since it does not sticks to stationary states from the outset. We have shown that nevertheless the latter naturally arise via Duistermaat–Heckman, and this somehow reconciles the two perspectives.

## 6. On the quantum measurement problem

The quantum measurement problem is actually the most tantalizing problem concerning the interpretation of quantum mechanics (we refer to [19,43] for a thorough discussion). In this section just make some remarks aiming at reinterpreting (part of) the treatment of the measurement problem given by Bohm in his (“orthodox”) book [9].

The upshot of his fairly detailed analysis (based on the Stern–Gerlach experiment and generalizations thereof) is that upon measuring an observable, say the energy  $H$ , a superposition of its eigenstates goes to a different superposition characterized by *uncontrollable* (relative) phase shifts (in view of the Heisenberg Uncertainty Principle):

$$\sum_j \alpha_j e_j \rightarrow \sum_j \alpha_j e^{i\beta_j} e_j. \tag{6.1}$$

We take, for definiteness,  $\alpha_j \neq 0$  for all  $j$ , and, as before, we may arrange things so that  $\beta_0 = 0$ . We consider the Schrödinger–von Neumann quantum evolution as taking place on the space of density matrices (mixed states), which may be identified, up to an  $i$  factor or, with a submanifold of  $u(V)$ . More explicitly, let  $\varrho$  be a density matrix, i.e. a positive operator ( $\varrho \geq 0$ ) on  $V$  with  $\text{Tr } \varrho = 1$ . Its evolution is governed by von Neumann's equation

$$\frac{\partial \varrho}{\partial t} = -i[H, \varrho] \quad (6.2)$$

which, when applied to a pure state (cf. (2.2)), reproduces the (projective form of the) Schrödinger equation (3.4); furthermore, in the notation of Section 3, see also above, one has  $\varrho = |v\rangle\langle v| = (\bar{\alpha}_i \alpha_j)$ .

Geometrically, the above equation says that the (undisturbed) evolution of a density matrix takes place on a  $U(V)$ -coadjoint orbit (with the customary identification of adjoint and coadjoint action via, e.g. the Killing metric on  $u(V)$  and up to an  $i$  factor), which is a symplectic manifold. This picture can be naturally supplemented by a  $C^*$ -algebraic one: indeed, the density matrices constitute precisely the *state space* of the finite dimensional  $C^*$ -algebra  $B(V)$  consisting of all linear operators on the finite dimensional space  $V$  (so they are necessarily bounded), see e.g. [29]. This space is closed under *convex* combinations, and this will be crucial for what follows.

Now, roughly speaking for the moment, the point is that, upon averaging over the phases (i.e. over a (long) series of measurements), one gets a diagonal density matrix  $\rho := (|\alpha_j|^2 \delta_{ij})$  giving rise to a statistical ensemble in which an assembly of equal systems is partitioned in subsystems with energy values  $\lambda_j$  in proportions  $|\alpha_j|^2$ . In view of the classical interpretation of the quantum formalism outlined above, we can rephrase the preceding description by saying that the measurement process gives rise to an *adiabatic* perturbation (since the action variables, i.e. the transition probabilities, do not change); hence, as in perturbation theory in classical mechanics, one averages over the “fast” (i.e. angle) variables, namely, over an  $n$ -dimensional torus (since a global phase change yields nothing), this boiling down to the mixed state above. More precisely, we may state the following kind of *averaging theorem* (cf. [3]) (valid in the non-degenerate case), whose proof is straightforward.

**Theorem 6.1.** *In terms of density matrices, the following formula holds*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T e^{-iHt} \cdot [(\bar{\alpha}_i \alpha_j)] dt = \int_G g \cdot [(\bar{\alpha}_i \alpha_j)] dg = (|\alpha_j|^2 \delta_{ij}) = \rho, \quad (6.3)$$

where  $e^{-iHt}$  denotes the standard Schrödinger evolution,  $g$  stands for the toral action (3.9), whereas  $dg$  again denotes the normalized Haar measure on  $G$ . Notice that both integrals make sense since they both represent generalized convex combinations of (pure) states, so they still define density matrices.

This “phase wash-out” or “decoherence” (see e.g. [19,43]) can be described geometrically by saying that the torus action determines a transition from the pure state  $U(V)$ -coadjoint orbit given by  $P(V)$  to the (mixed state) one labeled by  $\rho$ . The Hamiltonian, clearly, does not change.

Next we would like to present a geometric description of the “collapse of the wave function”, which should supplement the preceding mechanism, in terms of basic notions from geometric invariant theory [21,33]. We begin with a brief digression in order to clarify our perspective. Consideration of vector fields  $\eta^\sharp = J\xi^\sharp$ , with  $\xi^\sharp$  a fundamental vector field associated to a Hamiltonian compact Lie group  $G$ -action on a Kähler manifold  $X$  (with Lie algebra  $\mathfrak{g}$ ) is customary in geometric invariant theory, whereby such an action is extended to the complexification  $G^c$ , with Lie algebra  $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$ . Of course, such an extended action does not preserve the metric any longer. Under fairly general conditions (see [22]) one has the identification between Marsden–Weinstein and Mumford quotients, respectively:

$$\frac{X_0}{G} \cong \frac{X_s}{G^c} \tag{6.4}$$

with  $X_0 = \mu^{-1}(0)$ ,  $X_s := G^c \cdot X_0$  (the stable points in Mumford’s sense, see e.g. [21,23]). In our case we have  $X = P(V)$ ,  $G = \mathbf{T}^n$ ,  $\mathfrak{g} = i\mathbf{R}^n$ ,  $G^c \cong (\mathbf{C}^*)^n$ ,  $\mathfrak{g}^c = i\mathbf{R}^n \oplus \mathbf{R}^n$ ,  $\mu([v]) = (I_1, I_2, \dots, I_n)$  (here  $\mu$  denotes the toral moment map naturally induced from the  $u(V)$  one, cf. Sections 2 and 3, up to a scalar) and the above quotients are both reduced to the point  $[e_0]$ . The vertices of the polytope also correspond to the absolute minima ( $[e_0]$ ) and maxima ( $[e_j]$ ,  $j = 1, 2, \dots, n$ ) of the norm square of the toral moment map  $\mu$ . The slightly asymmetrical role of the critical points  $[e_j]$  just stems from our initial conventions. The action of the complex torus is no longer unitary (it is indeed a Lie subgroup of the full linear group  $GL(V)$ ).

In view of formulae (2.5) and (3.7) applied to  $P_j$  (complexification of  $-iP_j$ ) we get the following geometric portrait: upon measurement of the energy  $H = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_j \lambda_j P_j$  (we always require non-degeneracy of the energy levels), the system undergoes a gradient flow motion (with respect to the Fubini–Study metric) starting from an initial state  $[v]$  with velocity field  $P_j^\sharp$ , this of course with probability  $I_j = |\alpha_j|^2$ ; the velocity diminishes by gradual loss of uncertainty provided by the measurement until in the limit  $t \rightarrow +\infty$ , one gets for the energy the exact value  $\lambda_j$ , corresponding to the critical point  $[e_j]$  of the Hamiltonian.

It is indeed easy to check that, under the evolution  $[v] \mapsto e^{tP_j} \cdot [v]$ , one has, provided  $\alpha_j \neq 0$

$$\lim_{t \rightarrow +\infty} e^{tP_j} \cdot [v] = [e_j], \tag{6.5}$$

yielding the desired collapse, or reduction, of the superposition  $[v]$  to the stationary state  $[e_j]$ .

The (“dissipative”) process in question involves a violation of unitarity—this is mathematically clear, as we have seen, and it is physically reasonable as well, since we discuss the system evolution alone, neglecting both the measuring apparatus and the environment, cf. [19]—but linearity is retained. Resorting to the geometric picture of the orbit space, we may also say that the collapse of the wave function consists, geometrically, in a point in the polytope being “forced”, via (6.5), onto one of its vertices, with probabilities given by its  $\mathbf{R}^n$ -coordinates. The origin corresponds to the critical point  $[e_0]$ . Also, during the process, adiabaticity (action invariance, i.e. probability conservation) is clearly destroyed.

We stress the fact that our geometric picture should be seen as a (possibly useful) description, not as a “realistic” explanation. On the other hand, various mechanisms of dissipation

have been invoked in the physical literature (see [19] for a thorough up-to-date discussion) in connection with the collapse of the wave function. Among these, the idea of relaxing unitarity whilst keeping linearity is also present. Geometric invariant theory possibly makes this mathematically natural.

## 7. Second quantization as Bohr–Sommerfeld quantization

In this section we discuss another implication of complete integrability. In [4,38] it is observed that (geometric) quantization of a quantum mechanical system looked upon classically yields its second quantization. We comment on this as follows: having realized a (finite dimensional) quantum mechanical system as a classically completely integrable system (with the Riemannian structure giving the extra “quantum” ingredient) formally resembling a collection of classical harmonic oscillators (with a norm constraint)—this is clear from Section 3, but see also, e.g. [25]—we may wish to quantize it, for instance, via Bohr–Sommerfeld quantization (ignoring the Maslov correction for the moment, see e.g. [44]): we proceed as follows: first recall the formula for the classical Hamiltonian  $h$  (for  $\|v\| = 1$  and  $\lambda_0 = 0$ ):

$$h([v]) = \langle v | H v \rangle = \sum_{j=1}^n \lambda_j I_j. \quad (7.1)$$

Now, Bohr–Sommerfeld quantization requires, in our case:

$$I_j = n_j \in \mathbf{N}, \quad j = 1, 2, \dots, n, \quad (7.2)$$

giving rise to the (non-negative) energy levels

$$H(\{n_j\}) = \sum_{j=1}^n \lambda_j n_j. \quad (7.3)$$

Taking into account the bounds  $0 \leq I_j \leq 1$ ,  $j = 1, 2, \dots, n$ , this is possible if and only if  $n_j = 0$  for all  $j$ 's or  $n_j = \delta_{jk}$  for some  $k$ . That is we exactly recover the eigenstates and energy level of the initial system (the vertices of the moment map polytope). This is the simplest instance of a general result [45] establishing (equivariant, with respect to a toral action) equivalence of Bohr–Sommerfeld and holomorphic quantization.

However, upon removing the above constraints we get precisely the (bosonic) second quantization prescription (with the  $n_j$ 's becoming occupation numbers). Taking Maslov's correction into due account would yield the zero point energy contribution (cf. [21]), which is discarded in the infinite dimensional situation. Hence, we summarize the preceding discussion by saying that *second quantization can be interpreted as a kind of Bohr–Sommerfeld quantization of a quantum mechanical system looked upon classically*.

Moreover one can, by resorting, e.g. to [18], realize the (bosonic) second quantization scheme geometrically upon considering direct sums of tensor products of the hyperplane section bundle  $\mathcal{O}(1)$  on  $P(V)$  (whose holomorphic sections yield an  $n + 1$ -dimensional

complex vector space  $=: \Gamma$ , cf. [20]) and defining the symmetric Fock space  $\mathcal{F}$  as

$$\mathcal{F} := \bigoplus_{k=0}^{\infty} \Gamma^k \quad (7.4)$$

(symmetric tensor product understood, and obviously taking  $\Gamma^0 = \mathbf{C}$ ).

Notice that the above holomorphic section realization of the quantum Hilbert space is just a particular case of the coherent state formalism (in the geometric quantization framework via the Borel–Weil theorem). Roughly speaking, it just describes the quantum Hilbert space via probability amplitudes ( $|v\rangle \mapsto \langle v| \equiv |w\rangle \mapsto \langle v|w\rangle$ ). Incidentally, this would essentially yield compatibility between geometric quantization (applied to  $P(V)$ ) and geometric quantum mechanics. The literature concerning these topics is enormous, so, since we are not going to delve further into these problems, we just refer, particularly for the geometric aspects to [36,37,44] and also to [40,41], and references therein.

## 8. Conclusions and outlook

Our geometric approach is basically finite dimensional. However, this is far from being devoid of physical significance: indeed, one often works with a finite dimensional approximation, namely in quantum chemistry (Hartree–Fock), see e.g. [22]; another important example is provided by quantum computation, see e.g. [28]. The theory outlined above can be extended partially to the infinite dimensional case with few modifications (see however [16] as well, for a more general approach), provided  $H$  has a multiplicity-free semibounded discrete spectrum. One has the action of an infinite dimensional torus (still compact) and an infinite number of first integrals in involution. This is possibly the *simplest* example of an infinite dimensional integrable system. The above condition clearly excludes fundamental examples such as a free particle or unbound states of an electron. The Duistermaat–Heckman approach a priori fails. However, one may hope for an extension of [Theorem 6.1](#). via the measure theoretic techniques of [16].

The full implications of complete integrability are not clear to us at the moment. However, it may possibly have nothing to do a priori with *quantization* of classically integrable or chaotic systems: in our treatment the quantum system and Hamiltonian are given, and they do not necessarily come from a quantization procedure applied to some classical dynamical system (in particular, we do not tackle the crucial and difficult problem of determining or approximating the energy spectrum). Let us just observe, in passing, that the action of the “Bohmian” tori we have been considering throughout the paper destroys coherence of the wave function, so, in principle, they do not pull-back to tori on a, say, coherent state manifold (coming from geometrically quantizing a classical dynamical system).

Also, a deeper geometrical insight may prove useful in further penetrating the “mysteries” of the quantum measurement process and of quantum entanglement (see e.g. [12] for an interesting geometrical approach to the latter).

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